

EXERCISES for Calculus 2 (BT) - Chapter 1: Multivariable Calculus

Dr Janet Harris, Semester 1, 2012/13.

- For the given functions, state the domain and range and evaluate $f(1, 0)$ and $f(-1, 1)$.
 (a) $f(x, y) = |x - y|$, (b) $f(x, y) = x^2 e^{2xy}$, (c) $f(x, y) = \sqrt{4 - x^2 - y^2}$.
- For the following functions find all the first partial derivatives and evaluate them at the given point.

(a) $f(x, y) = x^3 y^2 - 1$, $(1, 2)$ (b) $z(x, y) = \frac{x}{y} + y e^x$, $(0, 1)$ (c) $f(r, \theta) = \sin(\theta \sqrt{r})$, $(4, \pi/3)$

(d) $z(x, y) = \tan^{-1}(\frac{y}{x})$, $(2, 1)$ (e) $g(x, y, z) = \frac{xz}{y + z}$, $(1, 2, 3)$.

- Find all the first order partial derivatives of the given functions.

(a) $z = \sqrt{1 + x^2 y^3}$, (b) $W(E, T) = e^{-E/kT}$ (k constant), (c) $w = \ln(x^2 y + z)$.

- Show that the given function satisfies the given differential equation:

$$z = \frac{x + y}{x - y}, \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$$

- The Wind Chill Index $I(T, v)$ is the perceived temperature at actual air temperature T ($^{\circ}\text{C}$) with wind speed v (km/hour). Using the table below, estimate the values of $I_T(12, 20)$ and $I_v(12, 20)$ and explain their practical significance.

		Wind speed (km/h)									
Actual temperature (°C)	$T \backslash v$	10	20	30	40	50	60	70	80	90	100
	20	18	16	14	13	13	12	12	12	12	12
	16	14	11	9	7	7	6	6	5	5	5
	12	9	5	3	1	0	0	−1	−1	−1	−1
	8	5	0	−3	−5	−6	−7	−7	−8	−8	−8

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by
 (a) $x^2 - y^2 + z^2 = 2x(y + z)$, (b) $xy + z + yz^2 = e^{xy}$, (c) $y \sin(z - 2y) = 1 + z \cos(2xy)$.
- Find all the second partial derivatives of $f(x, y) = 1 + x^2 \ln y$.
- Verify Clairaut's Theorem for the function

$$f(x, r) = \frac{e^{-xr}}{r}.$$

- Given $f(x, y, z) = x^2 e^{3z} \ln y$, (a) show that $f_{xyz} = f_{zyx}$, (b) find f_{xyyz} .
- Given $z = e^{xy}$, find $\frac{\partial^3 z}{\partial y \partial x^2}$.

11. Show that $f(x, y) = e^x \sin y$ satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

12. For 1 mole of an ideal gas, the absolute temperature T , pressure P and volume V are related by the equation $PV = RT$ where R is a constant. Show that

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1.$$

13-16. Find and classify the critical points of the following functions:

13. $f(x, y) = x^2 - \frac{1}{3}y^2 + 4x - 2y + 5$, 14. $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - x^2 - y^2$.
 15. $f(x, y) = x^4 + y^2 - 4xy$, 16. $f(x, y) = e^y - ye^x$.

17. Find positive numbers a, b and c such that $a + b + c = 36$ and abc is a maximum.

18. Find the absolute maximum and minimum of the function $f(x, y) = 2x^2 - y^2 + 6y$ on the disc $x^2 + y^2 \leq 16$. [Hint: Note that the boundary is a circle.]

19. Find the linearisation of the function $f(x, y) = ye^{xy}$ at the point $(0, 1)$ and use it to find the approximate value of $f(-0.1, 1.1)$. What is the percentage error in this result?

20. A child is measured to have height 1.38 metres and weigh 39 kg. Calculate their BMI (Body Mass Index - see slide 7). If these measurements have possible errors of ± 0.5 cm and ± 0.4 kg respectively, use differentials to estimate the maximum possible error in this BMI.

21. The dimensions of a rectangular box are measured to be 20 cm, 30 cm and 40 cm, each with a possible error of 0.2 cm. Use differentials to estimate the maximum error and percentage error in calculating the total surface area of the box.

22. A rectangular swimming pool measuring 20 m by 30 m is filled with water. The depth of water is measured at 5 m intervals in both directions, starting from one corner of the pool. The values (in meters) are shown in the table below. Estimate the volume of water in the pool. (Explain your method.)

$x \backslash y$	0	5	10	15	20	25	30
0	0.8	0.8	0.8	0.8	0.8	0.8	0.8
5	0.9	1.1	1.2	1.2	1.0	0.8	0.8
10	1.0	1.8	2.2	2.2	1.8	1.2	0.8
15	1.1	2.2	2.6	3.0	2.4	1.4	0.8
20	1.2	2.2	3.0	3.0	2.6	1.8	0.8

23. Evaluate the following integrals:

(a) $\int_0^1 \int_1^2 1 - x + \sqrt{y} \, dx \, dy$, (b) $\int_0^a \int_0^b x^2 y^3 \, dy \, dx$

(c) $\iint_R \frac{1}{(2x + y)^2} \, dA$, $R = [0, 1] \times [1, 3]$

(d) $\iiint_B 6x^3 y^2 z \, dV$, $B = \{(x, y, z) : 0 \leq x \leq 1, -1 \leq y \leq 2, 1 \leq z \leq 3\}$

24. Find the volume of the region lying under the surface $z = 6 - x - y$ and above the rectangular region $R = [0, 2] \times [2, 4]$.

25. Evaluate the following integrals:

$$(a) \int_0^{\pi/2} \int_0^{\cos x} \sin x \, dy \, dx, \quad (b) \int_0^{\sqrt{2}} \int_0^y y^2 e^{xy} \, dx \, dy$$

$$(c) \iint_D x^2 y^4 \, dA, \quad D = \{(x, y) : -y \leq x \leq y, \, 0 \leq y \leq 2\}$$

$$(d) \iint_R \cos(2x + y) \, dA, \quad R = \{(x, y) : 0 \leq x \leq \pi/2, \, -\pi \leq y \leq 2x\}$$

26. (a) Find $\iint_R x \cos y \, dA$ where R is the finite region in the first quadrant bounded by the curve $y = 1 - x^2$ and the coordinate axes.

(b) Find $\iint_T x \sqrt{y} \, dA$ where T is the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1, 2)$.

27. (a) Find the volume of the solid lying under the surface $z = 1/(x + y)$ and above the region in the xy -plane bounded by $x = 1$, $x = 2$, $y = 0$ and $y = x$.

(b) Find the volume under the surface $z = xy$ above the region of the xy -plane bounded by the curves $y = 2x$ and $y = x^2$.

28. For the given integral, (i) sketch the region of integration, (ii) rewrite the integral with the order of integration reversed, and thus (iii) evaluate the integral:

$$(a) \int_0^1 \int_{\sqrt{y}}^1 e^{x^3} \, dx \, dy, \quad (b) \int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx.$$